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A CUP PRODUCT PAIRING AND TIME-DUALITY FOR DISCRETE DYNAMICAL SYSTEMS

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With each isolating block for a homeomorphism of an orientable manifold we associate a cup product pairing. Up to an isomorphism, this is a pairing of the discrete Conley indices of the homeomorphism and its inverse on the isolating block. We show that this pairing is nondegenerate. As a corollary we prove a time duality result for the discrete Conley index.

0. INTRODUCTION

The observation that the Poincare–Lefschetz duality theorem can be used to establish a relationship between the Conley indices of an isolated invariant set with respect to forward and backward time comes from McCord [4]. A simple proof of a result of this type in the continuous setting was given by Mrozek and Srzednicki in [7]. In the unpublished typescript [9] a discrete counterpart of the time duality is proved. Roughly speaking, it states that the q-dimensional cohomological Conley index of an isolated invariant set with respect to an orientation preserving homeomorphism is equal to the $(n-q)$-dimensional homological Conley index of the same invariant set with respect to its inverse, where n is the dimension of the manifold. If the homeomorphism reverses orientation, the index maps with respect to forward and backward time differ by a sign. However, in applications another version of the time duality turns out to be of more use. Intersection pairings defined by Kurland [3], provide the language for a concise representation of the time-duality isomorphisms, which can be obtained from a statement about nondegeneracy of the intersection pairings on Conley indices. In our paper we follow this approach and prove a nondegeneracy theorem for a pairing of the discrete-time Conley indices defined by the cup product. In this respect, this paper extends the results of [9]. Our main motivation for this extension is an application for a topological version of the Smale–Birkhoff theorem [2].

1. LINEAR ALGEBRAIC PRELIMINARIES

Let F be a fixed field and V a vector space over F. For an endomorphism φ of V, the generalized image and the generalized kernel of φ are defined by

$$\operatorname{gim}\varphi = \bigcap_{k\in\mathbb{Z}^+} \operatorname{im}\varphi^k, \qquad \operatorname{gker}\varphi = \bigcup_{k\in\mathbb{Z}^+} \ker\varphi^k.$$

One can easily check that φ maps both $\operatorname{gim}\varphi$ and $\operatorname{gker}\varphi$ into themselves. Therefore, it induces endomorphisms of $V/\operatorname{gker}\varphi$ and $\operatorname{gim}\varphi$, which will be denoted by $L_1(\varphi)$ and $L_2(\varphi)$, respectively. The Leray reduction of φ [6, Section 4] is the endomorphism $L_2(L_1(\varphi))$ of $\operatorname{gim} L_1(\varphi)$. In the sequel, we shall denote it by $L(\varphi)$. It can be shown (see [6]) that the Leray reduction of any endomorphism is an isomorphism. We say that two endomorphisms $\varphi\colon V \to V$ and $\psi\colon W \to W$ are conjugate iff there exists an isomorphism $\alpha\colon V \to W$ (called a conjugacy) such that $\psi \circ \alpha = \alpha \circ \varphi$. Clearly, the conjugacy relation, which will be denoted by $\equiv$, is an equivalence relation. For an endomorphism φ by $[\varphi]$ we shall denote the conjugacy class of φ, i.e. the class of all endomorphisms which are conjugate to φ.

The endomorphism φ is called of finite type if there exists an $n \in \mathbb{N}$ such that $\dim \operatorname{im} \varphi^n < \infty$. Equivalently, φ is of finite type iff, for some $m \in \mathbb{N}$,

$$\operatorname{gim} \varphi = \operatorname{im} \varphi^m$$

and $\operatorname{gim} \varphi$ is finite dimensional.

Let us prove a few useful facts now.

PROPOSITION 1.1. If φ *is of finite type then it maps its generalized image isomorphically onto itself.*

Proof. Let $n \in \mathbb{N}$ be such that $\operatorname{gim} \varphi = \operatorname{im} \varphi^n$. Then

$$\operatorname{gim} \varphi = \operatorname{im} \varphi^{n+1} = \varphi(\operatorname{im} \varphi^n) = \varphi(\operatorname{gim} \varphi). \qquad \square$$

LEMMA 1.1. *Assume that* $\psi\colon W \to W$, $\varphi\colon V \to V$, $\alpha\colon V \to W$ *and* $\beta\colon W \to V$ *are homomorphisms such that the following diagram commutes*:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\alpha"] & V \arrow[d, "\alpha"] \\ W \arrow[r, "\psi"] \arrow[ur, "\beta"] & W \end{tikzcd}$$

Then:

(i) $L(\psi) \equiv L(\varphi)$,
(ii) ψ *is of finite type if and only if* φ *is of finite type*,
(iii) *if* ψ *and* φ *are of finite type then* α *maps* $\operatorname{gim} \varphi$ *isomorphically onto* $\operatorname{gim} \psi$ *and* β *maps* $\operatorname{gim} \psi$ *isomorphically onto* $\operatorname{gim} \varphi$.

Proof. For the proof of (i), see [6, Theorem 4.4].

In order to prove (ii), assume that φ is of finite type. There is an $n \in \mathbb{N}$ such that $\dim \operatorname{im} \varphi^n < \infty$. Since

$$\psi^{n+1} = (\alpha \circ \beta)^{n+1} = \alpha \circ (\beta \circ \alpha)^n \circ \beta = \alpha \circ \varphi^n \circ \beta$$

the image of ψ^{n+1} is finite dimensional. It follows that ψ is of finite type. Similarly, one proves that the reverse implication holds.

Since $\alpha \circ \varphi = \psi \circ \alpha$ and $\beta \circ \psi = \varphi \circ \beta$, $\alpha(\operatorname{gim} \varphi) \subset \operatorname{gim} \psi$ and $\beta(\operatorname{gim} \psi) \subset \operatorname{gim} \varphi$. Hence, by Proposition 1.1,

$$\operatorname{gim} \varphi = \varphi(\operatorname{gim} \varphi) = \beta \circ \alpha(\operatorname{gim} \varphi) \subset \beta(\operatorname{gim} \psi).$$

Analogously, one proves that $\operatorname{gim}\psi \subset \alpha(\operatorname{gim}\varphi)$. It follows that $\operatorname{gim}\varphi$ and $\operatorname{gim}\psi$ are of equal dimensions and hence the above inclusions imply (iii). $\square$

Corollary 1.1. *If an endomorphism $\varphi: V \to V$ is of finite type then $L(\varphi) \equiv L_2(\varphi)$.*

Proof. Let φ_i be the restriction of φ to $V_i = \operatorname{im}\varphi^i$ $(i \in \mathbb{Z}^+)$. We have the following diagram, in which the vertical arrows are inclusion-induced and the slant one is the appropriate restriction of φ:

$$\begin{tikzcd}
V_{i+1} \arrow[r, "\varphi_{i+1}"] \arrow[d] & V_{i+1} \arrow[d] \\
V_i \arrow[r, "\varphi_i"] \arrow[ur, "\varphi"] & V_i
\end{tikzcd}$$

By Lemma 1.1, $L(\varphi_i) \equiv L(\varphi_{i+1})$. Since φ is of finite type, $V_n = \operatorname{gim}\varphi$ for some n. Moreover, for such an n, φ_n is an isomorphism. Hence,

$$L(\varphi) = L(\varphi_0) \equiv L(\varphi_n) \equiv \varphi_n = L_2(\varphi)$$

and the proof is finished.

Lemma 1.2. *Let φ: $V \to V$ be an endomorphism. Assume that $\varphi^\star = \operatorname{Hom}(\varphi, F)$: $\operatorname{Hom}(V, F) \to \operatorname{Hom}(V, F)$ is of finite type. Then, φ is of finite type. Moreover, if $\xi \in \operatorname{gim}\varphi^\star$ is nonzero then there is an $x \in \operatorname{gim}\varphi$ such that $\xi(x) = 0$.*

Proof. Let $n \in \mathbb{N}$ be such that $\operatorname{im}\varphi^{\star n}$ is finite dimensional. Then, also $\operatorname{im}\varphi^n$ is finite dimensional and hence φ is of finite type. Take an $N \in \mathbb{N}$ be such that $\operatorname{im}\varphi^N = \operatorname{gim}\varphi$. There is an $\eta \in \operatorname{Hom}(V, F)$ such that $\xi = (\varphi^\star)^N(\eta) = \eta \circ \varphi^N$. Since ξ is nonzero, there exists a $y \in V$ such that

$$\xi(y) = \eta(\varphi^N(y)) \neq 0.$$

Now, $\varphi^N(y)$ belongs to $\operatorname{gim}\varphi$. Therefore, by Proposition 1.1, there is an $x \in \operatorname{gim}\varphi$ such that $\varphi^N(x) = \varphi^N(y)$. For such an x,

$$\xi(x) = \eta(\varphi^N(x)) = \eta(\varphi^N(y)) \neq 0. \qquad \square$$

In what follows, by a pairing we mean either a bilinear map defined on the Cartesian product of two vector spaces over F or the linear map corresponding to it defined on the tensor product of these spaces. A pairing $\omega: V_1 \times V_2 \to F$ is called nondegenerate iff for each nonzero $v \in V_1$ there exists a $w \in V_2$ such that $\omega(v, w) \neq 0$ and for each nonzero $w \in V_2$ there is a $v \in V_1$ such that $\omega(v, w) \neq 0$. The following proposition which we state for further reference is a restatement of a standard fact in linear algebra.

Proposition 1.2. *Let V_1 and V_2 be finite-dimensional vector spaces, a pairing $\omega: V_1 \times V_2 \to F$ be nondegenerate and $\alpha_i: V_i \to V_i$ $(i = 1, 2)$ be endomorphisms such that $\omega(\alpha_1(v), w) = \omega(v, \alpha_2(w))$ for each $v \in V_1$ and $w \in V_2$. Then $[\alpha_1] = [\operatorname{Hom}(\alpha_2, F)]$.*

Proof. Let $\beta: V_1 \to \mathrm{Hom}(V_2, F)$ be defined by $\beta(v) = \omega(v, .)$. By the nondegeneracy of ω, β is an isomorphism. Furthermore,

$$\beta(\alpha_1(v))(w) = \omega(\alpha_1(v), w) = \omega(v, \alpha_2(w)) = \beta(v)\,(\alpha_2(w)).$$

Hence $\beta \circ \alpha_1 = \mathrm{Hom}(\alpha_2, F) \circ \beta$, so that β is the desired conjugacy. □

2. THE DISCRETE CONLEY INDEX

In this section we recall briefly the basic concepts of the Conley index theory for homeomorphisms of locally compact metric spaces. For more details, the reader is referred to [6, 8] or [10]. Let X be a locally compact metric space and $f: X \to X$ be a homeomorphism. For a set $A \subset X$ the invariant part of A is defined by

$$\mathrm{Inv}_f A = \bigcap_{n \in \mathbb{Z}} f^n(A).$$

A compact set $K \subset X$ is called an isolating neighbourhood if $\mathrm{Inv}_f K \subset \mathrm{int}\, K$. A set S is an invariant set iff $f(S) = S$ or, equivalently, S is its own invariant part. It is said to be an isolated invariant set if and only if it is the invariant part of some isolating neighbourhood. A pair $Q = (Q_1, Q_0)$ is called an index pair for S iff the following three conditions hold:

(i) $\mathrm{Inv}_f\, \mathrm{cl}(Q_1 \backslash Q_0) = S \subset \mathrm{int}(Q_1 \backslash Q_0)$,
(ii) $f(Q_0) \cap Q_1 \subset Q_0$,
(iii) $f(Q_1 \backslash Q_0) \subset Q_1$.

For such a pair, the index map $f_Q: Q_1/Q_0 \to Q_1/Q_0$ is defined. More precisely, f_Q is the map induced by f, given by the formula

$$f_Q([x]) = \begin{cases} [f(x)] & \text{if } x, f(x) \in Q_1 \backslash Q_0 \\ [Q_0] & \text{otherwise.} \end{cases}$$

We note that, as it is usually done in the Conley index theory, by the quotient space Q_1/Q_0 we mean the *pointed* space resulting from Q_1 as the points of Q_0 are identified to a single distinguished point denoted by $[Q_0]$. The q-dimensional cohomological (homological) Conley index of f_Q, which will be denoted by $h^q(S, f, X)$ ($h_q(S, f, X)$), is the conjugacy class of the Leray reduction of the endomorphism $H^q(f_Q)$ of $H^q(Q_1/Q_0)$ (respectively, $H_q(f_Q)$ of $H_q(Q_1/Q_0)$). By H^* we mean the singular cohomology. In the sequel we are also going to use the Čech cohomology, which will be denoted by $\bar{H}^*$. Our reference for algebraic topology is [1].

3. THE CUP PRODUCT PAIRING

Throughout this and the next sections, F is a fixed field and f stands for a homeomorphism of an n-dimensional connected F-orientable topological manifold M into itself. All homology and cohomology used in the sequel has coefficients in F. Let $d(f)$ be $+1$ if f preserves orientation and -1 otherwise. By $d(f)$ we shall also denote the homomorphism of F being the multiplication by $d(f) = \pm 1$.

Let N be an isolating block for f, i.e. a compact subset of M such that $f^{-1}(N) \cap N \cap f(N) \subset \mathrm{int}(N)$. For $i \in \mathbb{Z}^+$ we define the sets N_i^+ and N_i^- inductively by

$$N_0^+ = N_0^- = \emptyset$$

$$N_{i+1}^+ = N \cap f^{-1}((M \backslash N) \cup N_i^+)$$

$$N_{i+1}^- = N \cap f((M \backslash N) \cup N_i^-).$$

Thus, N_i^+ (N_i^-) is the set of all points of N which are mapped outside N by a jth iterate of f (respectively, f^{-1}), for some $j \in \{1, 2, \dots, i\}$. By $\bar{N}_i^+$ and $\bar{N}_i^-$ we denote the closures of the sets N_i^+ and N_i^- (respectively). We note that the $N_i^\pm$ sets are open in N and $\bar{N}_i^\pm$ are compact. Moreover, for each $i \in \mathbb{Z}^+$, $N_i^\pm \subset \bar{N}_i^\pm \subset N_{i+1}^\pm$.

Definition 3.1. The graded endomorphisms

$$f_+^*: H^*(N, N_2^+) \to H^*(N, N_2^+)$$

and

$$f_-^*: H^*(N, N_2^-) \to H^*(N, N_2^-)$$

are defined in such a way that the following two diagrams commute:

$$\begin{tikzcd}
H^*(N, N_2^+) \arrow[dd, "f_+^*"] \arrow[dr, "f^*"] & \\
& H^*(N\backslash N_1^+, N_2^+\backslash N_1^+) \\
H^*(N, N_2^+) \arrow[ur, "\overset{\cong}{\mathrm{exc}}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
H^*(N, N_2^-) \arrow[dd, "f_-^*"] \arrow[dr, "(f^{-1})^*"] & \\
& H^*(N\backslash N_1^-, N_2^-\backslash N_1^-) \\
H^*(N, N_2^-) \arrow[ur, "\overset{\cong}{\mathrm{exc}}"'] &
\end{tikzcd}$$

The graded subspaces $R^*(N, N_2^+)$ of $H^*(N, N_2^+)$ and $R^*(N, N_2^-)$ of $H^*(N, N_2^-)$ are defined by

$$R^*(N, N_2^\pm) = \mathrm{gim} f_\pm^*.$$

By $g_\pm^*$ we denote the endomorphisms of $R^*(N, N_2^\pm)$ being the restrictions of $f_\pm^*$.

Now, we can define the $\cup$-product pairing. In the definition below, $\langle . , . \rangle$ stands for the scalar product [1, section VII.1] and, for a compact set $K \subset M$, by $o_K \in H_n(M, M\backslash K)$ we mean the fundamental class of M along K [1, Definition VIII.4.1].

Definition 3.2. The $\cup$-product pairing

$$\{.,.\}: (R^*(N, N_2^+) \otimes R^*(N, N_2^-))_n \to F$$

is defined to be the following composition:

$$(R^*(N, N_2^+) \otimes R^*(N, N_2^-))_n \longrightarrow (H^*(N, N_2^+) \otimes H^*(N, N_2^-))_n$$

$$\xrightarrow{\ \cup\ } H^n(N, N_2^+ \cup N_2^-) \xleftarrow[\mathrm{exc}]{\cong} H^n(M, (M\backslash N) \cup N_2^+ \cup N_2^-)$$

$$\xrightarrow{\ \langle \cdot, o_{N\backslash(N_2^+ \cup N_2^-)} \rangle\ } F$$

Note that in the above definition an unmarked arrow represents an appropriate inclusion-induced homomorphism (this convention applies to all diagrams in this paper).

The main result of this section is the following theorem, stating that g_+^* and g_-^* are dual with respect to the $\cup$-product pairing.

THEOREM 3.1. *If* g_-^* *is an isomorphism then, for each* $q \in \mathbb{Z}$, $\xi \in R^q(N, N_2^+)$ *and* $\eta \in R^{n-q}(N, N_2^-)$,

$$\{g_+^*(\xi), \eta\} = d(f)\{\xi, g_-^*(\eta)\}.$$

Proof. The idea is quite straighforward: we use the naturality of cup and scalar products with respect to inclusions and the map f. The only problem is to choose interfacing pairs allowing to relate g_+^* and g_-^*.

We have the following commutative diagram:

$$\begin{tikzcd}
H^*(N, N_2^-) \arrow[d, "\overset{\cong}{\mathrm{exc}}"] \arrow[dr, "f^*"] & \\
H^*(N \backslash N_1^-, N_2^- \backslash N_1^-) \arrow[r, "f^*"] & H^*(N \backslash N_1^+, N_1^- \backslash N_1^+) \\
H^*(N, N_2^-) \arrow[u, "(f^{-1})^*"] \arrow[ur] &
\end{tikzcd}$$

One can use this diagram and Definition 3.1 to obtain another one, namely,

$$\begin{tikzcd}
R^*(N, N_2^-) \arrow[r] \arrow[d, "(g_-^*)^{-1}"] & H^*(N, N_2^-) \arrow[dr, "f^*"] & \\
 & & H^*(N \backslash N_1^+, N_1^- \backslash N_1^+) \\
R^*(N, N_2^-) \arrow[r] & H^*(N, N_2^-) \arrow[uu, "f_-^*"] \arrow[ur] &
\end{tikzcd}$$

The above diagram together with Definition 3.1 implies the commutativity of the upper part of the following one (its remaining part is easily checked to commute as well):

$$\begin{tikzcd}[column sep=small]
R^q(N, N_2^+) \otimes R^{n-q}(N, N_2^-) \arrow[d] & & R^q(N, N_2^+) \otimes R^{n-q}(N, N_2^-) \arrow[ll, "g_+^* \otimes (g_-^*)^{-1}"'] \arrow[d] \\
H^q(N, N_2^+) \otimes H^{n-q}(N, N_2^-) \arrow[dr] \arrow[dd, "\cup"] & & H^q(N, N_2^+) \otimes H^{n-q}(N, N_2^-) \arrow[dl, "f^* \otimes f^*"] \arrow[dd, "\cup"'] \\
& H^q(N \backslash N_1^+, N_2^+ \backslash N_1^+) \otimes H^{n-q}(N \backslash N_1^+, N_1^- \backslash N_1^+) \arrow[d, "\cup"] & \\
H^n(N, N_2^+ \cup N_2^-) \arrow[r] & H^n(N \backslash N_1^+, (N_2^+ \cup N_1^-) \backslash N_1^+) & H^n(N, N_2^+ \otimes N_2^-) \arrow[l, "f^*"'] \\
H^n(M, (M \backslash N) \cup N_2^+ \cup N_2^-) \arrow[u, "\overset{\cong}{\mathrm{exc}}"'] \arrow[dr] \arrow[ddr, "\langle \cdot, o_{N \backslash (N_2^+ \cup N_2^-)} \rangle"'] & & H^n(M, (M \backslash N) \cup N_2^+ \cup N_2^-) \arrow[u, "\overset{\cong}{\mathrm{exc}}"'] \arrow[dl, "f^*"'] \arrow[dd, "\langle \cdot, o_{N \backslash (N_2^+ \cup N_2^-)} \rangle"] \\
& H^n(M, (M \backslash N) \cup N_2^+ \cup N_1^-) \arrow[uu, "\overset{\cong}{\mathrm{exc}}"'] \arrow[d, "\langle \cdot, o_{N \backslash (N_2^+ \cup N_1^-)} \rangle"] & \\
& F & F \arrow[l, "d(f)"]
\end{tikzcd}$$

We conclude that, for $\xi \in R^q(N, N_2^+)$ and $\eta \in R^{n-q}(N, N_2^-)$,

$$\{g_+^*(\xi), (g_-^*)^{-1}(\eta)\} = d(f)\{\xi, \eta\}$$

and the identity in the theorem follows. □

4. NONSINGULARITY OF THE CUP PRODUCT PAIRING

In what follows, ρ will stand for the natural transformation of the Čech cohomology functor into the singular cohomology functor, as defined in [1, Definition VIII. 6.11]. Thus, for each locally compact pair $A \subset B$ contained in some ENR and $q \in \mathbb{Z}$ we have a homomorphism $\rho = \rho_{(B, A)} : \bar{H}^q(B, A) \to H^q(B, A)$.

LEMMA 4.1. *Let N be an isolating block for f. Then f_+^* and f_-^* are of finite type* (*i.e. of finite type in any dimension*),

$$h^q(\mathrm{Inv}_f\, N, f, M) = [L(f_+^q)] = [g_+^q]$$

and

$$h^q(\mathrm{Inv}_f\, N, f^{-1}, M) = [L(f_-^q)] = [g_-^q].$$

Proof. First of all, notice that $Q = (N, \bar{N}_1^+)$ is an index pair for f (cf. [5, Proposition 4.7]). Let $\pi : (N, \bar{N}_1^+) \to N/\bar{N}_1^+$ be the projection map. We have the following commutative diagram, in which j_1^* and j_3^* are inclusion-induced. In its second row the convention of treating quotient spaces as pointed spaces is violated.

$$\begin{tikzcd}
H^q(N/\bar{N}_1^+) \arrow[rr, "f_Q^*"] \arrow[drr, "h^* = f_Q^*"] & & H^q(N/\bar{N}_1^+) \\
H^q(N/\bar{N}_1^+, \pi(\bar{N}_2^+)) \arrow[u, "j_1^*"] \arrow[r, "f_Q^*"] \arrow[d, "\cong"', "j_2^* = \pi^*"] & H^q(N/\bar{N}_1^+ \setminus [\bar{N}_1^+], \pi(\bar{N}_2^+) \setminus [\bar{N}_1^+]) \arrow[d, "j_4^* = \pi^*"] & H^q(N/\bar{N}_1^+, \pi(\bar{N}_2^+)) \arrow[l, "\cong"', "\mathrm{exc}"] \arrow[u, "j_1^*"'] \arrow[d, "\cong"', "j_2^*"] \\
H^q(N, \bar{N}_2^+) \arrow[r, "f^*"] \arrow[d, "\cong"', "\mathrm{id}"] & H^q(N \setminus \bar{N}_1^+, \bar{N}_2^+ \setminus \bar{N}_1^+) & H^q(N, \bar{N}_2^+) \arrow[l, "\cong"', "\mathrm{exc}"] \\
H^q(N, \bar{N}_2^+) \arrow[r, "f^*"] \arrow[d, "j_3^*"] & H^q(N \setminus N_1^+, \bar{N}_2^+ \setminus N_1^+) \arrow[u, "\cong", "\mathrm{exc}"'] \arrow[d] & H^q(N, \bar{N}_2^+) \arrow[l, "\cong"', "\mathrm{exc}"] \arrow[u, "\cong"', "\mathrm{id}"] \arrow[d] \\
H^q(N, N_2^+) \arrow[r, "f^*"] \arrow[ur, "f^*"] & H^q(N \setminus N_1^+, N_2^+ \setminus N_1^+) & H^q(N, N_2^+) \arrow[l, "\cong"', "\mathrm{exc}"]
\end{tikzcd}$$

Note that j_4 is a homeomorphism and therefore j_4^* is an isomorphism. From the diagram it follows that so is j_2^*. The homomorphism $f_Q^*: H^q(N/\bar{N}_1^+) \to H^q(N/\bar{N}_1^+)$ is of finite type by [11, Lemma 5.2] (as N is a compact subset of a manifold, it is contained in an open subset of M being an ENR; cf. [1, Proposition VIII.1.3], so that the argument of [11] works in our setting). It follows by Lemma 1.1 (ii) that so is f_+^q (it is the composition of the lower horizontal arrows). By the (i) part of the same Lemma and Corollary 1.1, the statements concerning the Conley index with respect to f hold as well. The remaining ones can be proved by time reversal. $\square$

THEOREM 4.1. *For any isolating block N for f and $q \in \mathbb{Z}$, the cup product pairing*

$$\{.,.\}: R^q(N, N_2^+) \otimes R^{n-q}(N, N_2^-) \to F$$

associated with N is nondegenerate.

Proof. Take any nonzero $\xi_0 \in R^{n-q}(N, N_2^-)$. Let $K_1 = (M \setminus \bar{N}_2^+) \cup N_1^+$, $K_2 = ((M \setminus N) \cup N_2^- \setminus \bar{N}_2^+) \cup N_1^+$, $L_1 = M \setminus \bar{N}_2^+$ and $L_2 = (M \setminus N) \cup N_2^- \setminus \bar{N}_2^+$. Figure 1 shows these sets (note that all of them are open) in the case of N being a rectangular isolating block for a hyperbolic fixed point. In all diagrams in this proof, by i with a subscript we denote an inclusion map.

Here is an outline of the proof. First, one finds a nonzero element ξ_1 in $H^{n-q}(L_1, L_2)$ which, in some natural sense, corresponds to ξ_0 and is not annihilated by any iterate of the endomorphism induced by f^{-1} (defined with the aid of the pair (K_1, K_2) in a similar way as f_-^*). Lemma 1.2 and the fact that the coefficient ring is a field allow to find an element η_0 of $H_{n-q}(L_1, L_2)$ whose scalar product with ξ_1 is nonzero and which belongs to the generalized image of the endomorphism induced by f^{-1}. Since $(M \setminus L_2, M \setminus L_1)$ is an index pair for f, the Alexander–Spanier dual of η_0 can be viewed as an element of $R^q(N, N_2^+)$. The final step is to show that the $\{.,.\}$ pairing of that element with ξ_0 is indeed nonzero.

Consider Fig. 2 below, in which the homomorphisms on the bottom are $\varphi = \mathrm{Hom}(H_{n-q}(i_5), F)$ and $\psi = \mathrm{Hom}(H_{n-q}(k), F)$ and the α homomorphisms are given by

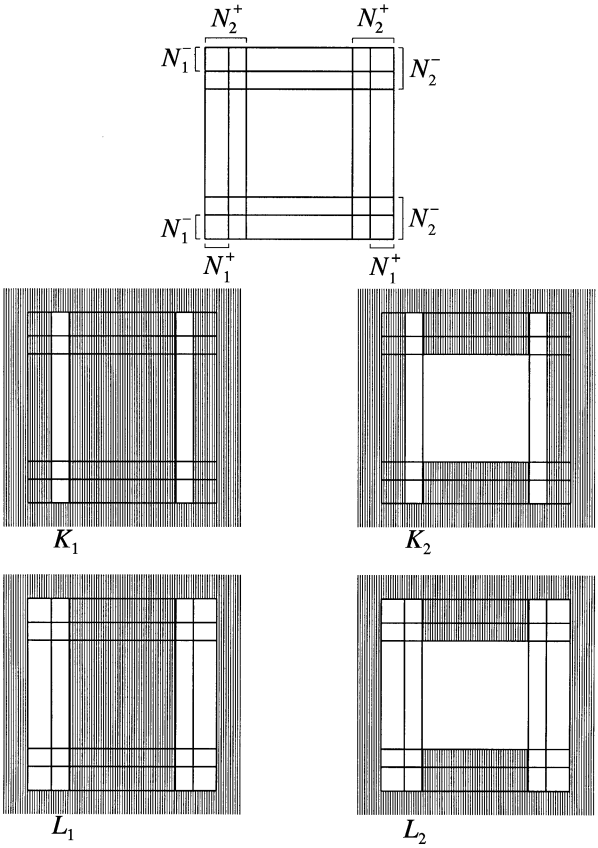

Fig. 1.

$\alpha(\xi)(\eta) = \langle \xi, \eta \rangle$ and are isomorphisms since the coefficient ring is a field [1, Proposition VII.1.7]. By Lemma 1.1, the composition $\alpha \circ i_5^* \circ (i_4^*)^{-1} \circ i_3^* \circ i_2^* \circ i_1^*$ maps $R^{n-q}(N, N_2^-)$ isomorphically onto $\operatorname{gim} \psi \circ \varphi^{-1}$. Therefore, $\alpha \circ i_5^* \circ (i_4^*)^{-1} \circ i_3^* \circ i_2^* \circ i_1^*(\xi_0)$ is a nonzero element of $\operatorname{gim} \psi \circ \varphi^{-1}$. By Lemma 1.2 (note that Fig. 2 and Lemma 4.1 show that $\psi \circ \varphi^{-1}$ is of finite type) there exists an $\eta_0 \in \operatorname{gim}(H_{n-q}(i_5)^{-1} \circ H_{n-q}(k))$ such that for $\xi_1 = i_5^* \circ (i_4^*)^{-1} \circ i_3^* \circ i_2^* \circ i_1^*(\xi_0)$,

$$\langle \xi_1, \eta_0 \rangle = \alpha(\xi_1)(\eta_0) \neq 0.$$

Let $M_i = (N \backslash N_i^-) \cup \bar{N}_2^+$ (for $i = 1, 2$). Consider the commutative up to a sign (this refers to the top left rectangle, only if f reverses orientation) Fig. 3. Note that its upper two rectangles commute by [1, VIII.7.6] and [1, Proposition VIII.10.10] and D_1 and D_2 are the Poincare–Lefschetz duality isomorphisms [1, Proposition VIII.7.2]. The arrows marked with ρ' represent homomorphisms being compositions of the natural transformation ρ and an inclusion map.

Let us take a look at the upper four levels of Fig. 3. Since $\psi \circ \varphi^{-1}$ is of finite type, Lemma 1.2 implies that so is $(i_{5*})^{-1} \circ k_*$. By Lemma 1.1, there is a $\xi_2 \in \operatorname{gim}((i_8^*)^{-1} \circ \psi')$ such that $D_1 \circ i_6^* \circ i_7^*(\xi_2) = \eta_0$. There are open neighbourhoods U, V of N and $\bar{N}_2^+$ such that $V \subset U$ and $\bar{\xi}_2 \in H^q(U, V)$ such that $\xi_2 = u(\bar{\xi}_2)$, where $u: H^q(U, V) \to \bar{H}^q(N, \bar{N}_2^+)$ is the universal transformation. To make the formulas simpler, let $j = i_1 \circ i_2 \circ i_3$. By [1, VIII.8.12],

$$0 \neq \langle \xi_1, i_6^* \circ i_7^*(\xi_2) \cap o_{M_2} \rangle = (-1)^{q(n-q)} \langle i_6^* \circ i_7^*(\xi_2) \cup \xi_1, o_{M_2} \rangle = \langle v(\bar{\xi}_2) \cup \xi_1, o_{M_2} \rangle,$$

$$
\begin{tikzcd}[column sep=small]
H^{n-q}(N, N_2^-) \arrow[r, "(f^{-1})^*"] \arrow[d, "i_1^*"'] & H^{n-q}(N\setminus N_1^-, N_2^-\setminus N_1^-) \arrow[dd] & H^{n-q}(N, N_2^-) \arrow[l, "\underset{\mathrm{exc}}{\cong}"'] \arrow[d] \\
H^{n-q}(N\setminus N_1^+, N_2^-\setminus N_1^+) \arrow[ur, "(f^{-1})^*"] \arrow[dr, "(f^{-1})^*"] \arrow[dd, "i_2^*"'] & & H^{n-q}(N\setminus N_1^+, N_2^-\setminus N_1^+) \arrow[dl, "\underset{\mathrm{exc}}{\cong}"'] \arrow[dd] \\
& H^{n-q}(N\setminus(N_1^+\cup N_1^-), N_2^-\setminus(N_1^-\cup N_1^+)) \arrow[dd] & \\
H^{n-q}(N\setminus N_2^+, N_2^-\setminus N_2^+) \arrow[ur, "(f^{-1})^*"] \arrow[dr, "(f^{-1})^*"] \arrow[dd, "i_3^*"'] & & H^{n-q}(N\setminus N_1^+, N_2^-\setminus N_2^+) \arrow[dl, "\underset{\mathrm{exc}}{\cong}"'] \arrow[dd] \\
& H^{n-q}(N\setminus(N_2^+\cup N_1^-), N_2^-\setminus(N_1^-\cup N_2^+)) \arrow[dd] & \\
H^{n-q}(N\setminus \bar N_2^+, N_2^-\setminus \bar N_2^+) \arrow[ur, "(f^{-1})^*"] \arrow[dr, "(f^{-1})^*"] & & H^{n-q}(N\setminus \bar N_2^+, N_2^-\setminus \bar N_2^+) \arrow[dl, "\underset{\mathrm{exc}}{\cong}"'] \\
& H^{n-q}(N\setminus(\bar N_2^+\cup N_1^-), N_2^-\setminus(N_1^-\cup \bar N_2^+)) & \\
H^{n-q}(K_1, K_2) \arrow[uu, "i_4^*"', "\underset{\mathrm{exc}}{\cong}"] \arrow[r, "(f^{-1})^*"] \arrow[d, "i_5^*"', "\underset{\mathrm{exc}}{\cong}"] & H^{n-q}(L_1, L_2) \arrow[u] & H^{n-q}(K_1, K_2) \arrow[l, "\underset{\mathrm{exc}}{\cong}"'] \arrow[uu, "\underset{\mathrm{exc}}{\cong}"] \arrow[d, "\underset{\mathrm{exc}}{\cong}"] \\
H^{n-q}(L_1, L_2) \arrow[d, "\cong"', "\alpha=\langle -,\cdot\rangle"] & H^{n-q}(K_1, K_2) \arrow[l, "\underset{\mathrm{exc}}{\cong}", "i_5^*"'] \arrow[ul, "\mathrm{id}"] \arrow[r, "k^*=(f^{-1})^*"] \arrow[d, "\cong"', "\alpha"] & H^{n-q}(L_1, L_2) \arrow[ul, "\mathrm{id}"'] \arrow[d, "\cong"', "\alpha"] \\
\mathrm{Hom}(H_{n-q}(L_1, L_2), F) & \mathrm{Hom}(H_{n-q}(K_1, K_2), F) \arrow[l, "\varphi", "\underset{\mathrm{exc}}{\cong}"'] \arrow[r, "\psi"] & \mathrm{Hom}(H_{n-q}(L_1, L_2), F)
\end{tikzcd}
$$

Fig. 2.

where $v: H^q(U, V) \to \bar{H}^q(M_2, \bar{N}_2^+)$ is the universal transformation. Let us note that the cup product used above is the one defined on the product of the Čech and singular cohomology [1, VIII.8.10]. Since $v(\bar{\xi}_2) \in \bar{H}^q(M_2, \bar{N}_2^+)$ and $\xi_1 \in H^{n-q}(L_1, L_2) = H^{n-q}(L_1, M\backslash M_2)$, $v(\bar{\xi}_2) \cup \xi_1 \in H^n(M, M\backslash M_2)$. The cup product maps on the next diagram are singular cohomology products.

In order to finish the proof, let us consider the commutative diagram (Fig. 4) in which α_1, α_2, γ_1, γ_2, γ_3, δ_1, δ_2, δ_3, δ_4, ι_0 and ι_1 are induced by the inclusion maps. By the commutativity of Figs 3 and 4 and the definition of the appropriate cup product [1, VIII.8.9],

$$
\begin{aligned}
0 \neq \langle v(\bar{\xi}_2) \cup \xi_1, o_{M_2} \rangle &= \langle \iota_0 \circ \delta_1^{-1} \circ \beta_1 \circ \alpha_1(\bar{\xi}_2 \otimes \xi_1), o_{M_2} \rangle \\
&= \langle \delta_1^{-1} \circ \beta_1 \circ \alpha_1(\bar{\xi}_2 \otimes \xi_1), o_{N\backslash(V \cup N_2^-)} \rangle \\
&= \langle \delta_2^{-1}(i_{11}^*(\bar{\xi}_2) \cup (i_4^* \circ (i_5^*)^{-1})\ (\xi_1)), o_{N\backslash(V \cup N_2^-)} \rangle \\
&= \langle \delta_2^{-1}(i_9^* \circ i_{12}^*(\bar{\xi}_2) \cup j^*(\xi_0)), o_{N\backslash(V \cup N_2^-)} \rangle \\
&= \langle \delta_3^{-1}(i_{12}^*(\bar{\xi}_2) \cup \xi_0), o_{N\backslash(V \cup N_2^-)} \rangle \\
&= \langle \delta_4^{-1}(i_{10}^* \circ i_{12}^*(\bar{\xi}_2) \cup \xi_0), o_{N\backslash(N_2^+ \cup N_2^-)} \rangle \\
&= \{i_{13}^*(\bar{\xi}_2), \xi_0\} = \{\rho' \circ u(\bar{\xi}_2), \xi_0\} = \{\rho'(\xi_2), \xi_0\}.
\end{aligned}
$$

Since $\xi_2 \in \mathrm{gim}\,(i_8^*)^{-1} \circ \psi'$, $\rho'(\xi_2) \in R^q(N, N_2^+)$ by the lower part of Fig. 3.

$$
\begin{tikzcd}
H_{n-q}(L_1, L_2) \arrow[rr, "k_* = (f^{-1})_*"] & & H_{n-q}(K_1, K_2) & H_{n-q}(L_1, L_2) \arrow[l, "i_{5*}"', "\underset{\text{exc}}{\cong}"] \\
\bar{H}^q(M_2, \bar{N}_2^+) \arrow[u, "\cong", "D_1 = \cap o_{M_2}"'] \arrow[rr, "f^*"] \arrow[rrd, "f^*"] & & \bar{H}^q(M_2 \setminus N_1^+, \bar{N}_2^+ \setminus N_1^+) \arrow[u, "\cong", "D_1 = \cap o_{M_2 \setminus N_1^+}"'] & \bar{H}^q(M_2, \bar{N}_2^+) \arrow[l, "\underset{\text{exc}}{\cong}"] \arrow[u, "\cong", "D_1"'] \\
\bar{H}^q(M_1, \bar{N}_2^+) \arrow[u, "i_6^*"'] \arrow[rr, "f^*"] \arrow[rrd, "f^*"] & & \bar{H}^q(M_1 \setminus N_1^+, \bar{N}_2^+ \setminus N_1^+) \arrow[u] & \bar{H}^q(M_1, \bar{N}_2^+) \arrow[l, "\underset{\text{exc}}{\cong}"] \arrow[u] \\
\bar{H}^q(N, \bar{N}_2^+) \arrow[u, "i_7^*"'] \arrow[rr, "\psi' = f^*"] \arrow[rd, "\rho'"] & & \bar{H}^q(N \setminus N_1^+, \bar{N}_2^+ \setminus N_1^+) \arrow[u] \arrow[d, "\rho'"] & \bar{H}^q(N, \bar{N}_2^+) \arrow[l, "i_8^*"', "\underset{\text{exc}}{\cong}"] \arrow[u] \arrow[d, "\rho'"] \\
 & H^q(N, N_2^+) \arrow[r, "f^*"] & H^q(N \setminus N_1^+, N_2^+ \setminus N_1^+) & H^q(N, N_2^+) \arrow[l, "\underset{\text{exc}}{\cong}"] \\
H^q(U, V) \arrow[uu, "u"] \arrow[ru, "i_{13}^*"] \arrow[rd, "i_{12}^*"'] \arrow[dd, "i_{11}^*"'] & & & \\
 & H^q(N, N \cap V) \arrow[uu, "i_{10}^*"'] \arrow[ld, "i_9^*", "\underset{\text{exc}}{\cong}"'] & & \\
H^q(N \setminus \bar{N}_2^+, N \cap V \setminus \bar{N}_2^+) & & &
\end{tikzcd}
$$

Fig. 3.

To sum up, we have proved that for each nonzero $\xi_0 \in R^{n-q}(N, N_2^-)$ there is a $\xi_0' \in R^q(N, N_2^+)$ such that $\{\xi_0', \xi_0\} \neq 0$. By time reversal one can show that also for each nonzero $v_0 \in R^q(N, N_2^+)$ there is an $v_0' \in R^{n-q}(N, N_2^-)$ such that $\{v_0, v_0'\} \neq 0$. The proof is finished. □

We finish with a statement of the time-duality theorem for discrete dynamical systems. For a conjugacy class $\Phi = [\varphi]$ and $k \in F$, by $\mathrm{Hom}(\Phi, F)$ and $k\Phi$ we denote the conjugacy classes of $\mathrm{Hom}(\varphi, F)$ and $k\varphi$. Note that these classes do not depend on the choice of a representative for Φ.

COROLLARY 4.1. *Let S be an isolated invariant set with respect to a homeomorphism f of an n-dimensional F-orientable manifold M. Then*

$$d(f)\; h^q(S, f, M) = \mathrm{Hom}(h^{n-q}(S, f^{-1}, M), F) \tag{4.1}$$

and

$$h_q(S, f, M) = d(f) h^{n-q}(S, f^{-1}, M). \tag{4.2}$$

Proof. Since there exists an isolating block for S (the argument of [8, Theorem 4.5] can be used to obtain a simple proof), (4.1) follows from Proposition 1.2, Lemma 4.1 and Theorems 3.1 and 4.1. Application of the universal coefficient formula yields (4.2). □

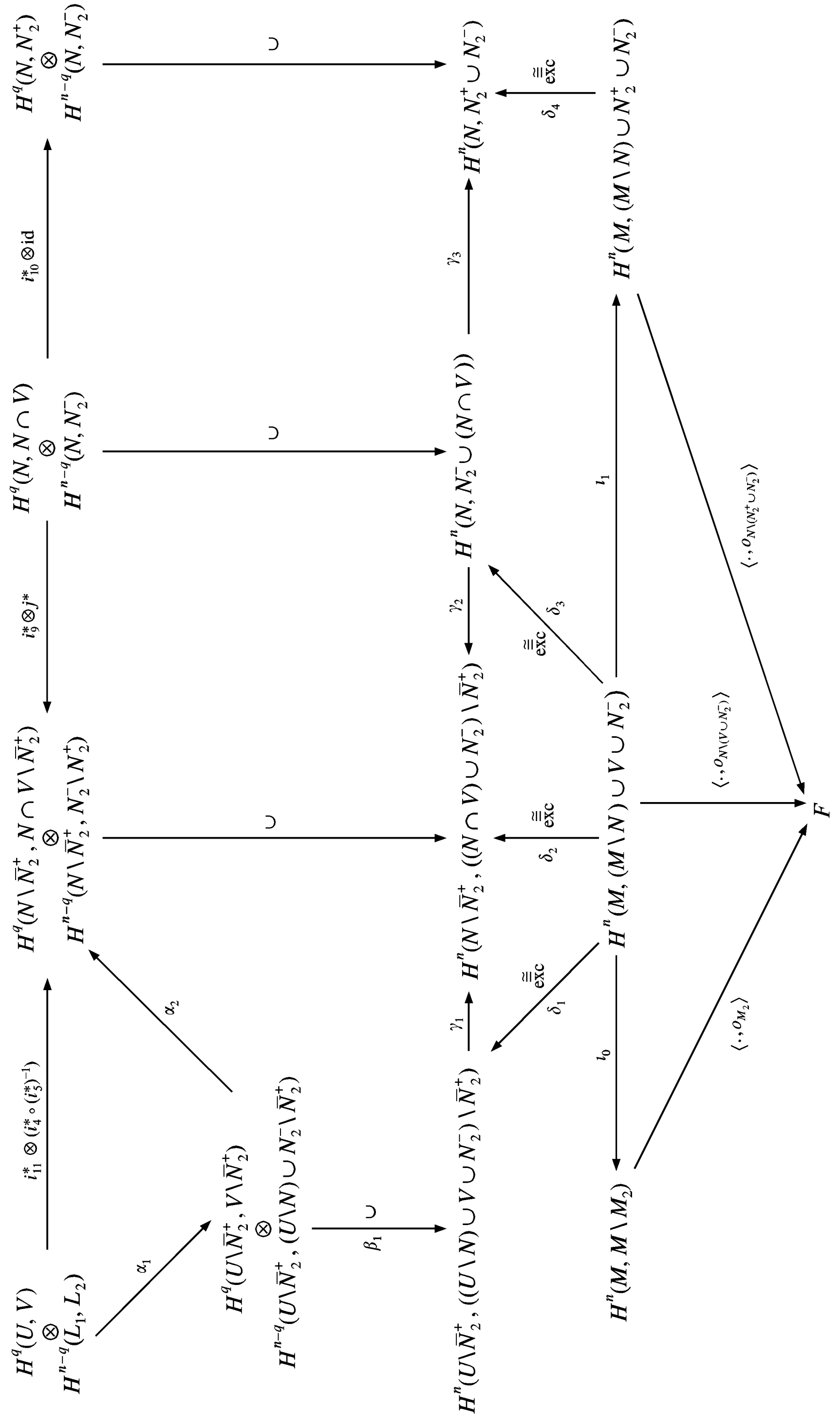

Fig. 4.

Let us note that, in fact, (4.2) does not require field coefficients—it can be proved directly (not as a corollary of the nondegeneracy of the cup product pairing) by a similar method (see also [9]). An application of the above theorem for f being the time-one map for a continuous-time flow yields continuous-time duality of [4, 7].

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